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TREE PRUNING AND LATTICE STATISTICS ON BETHE LATTICES

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Abstract

The method of tree pruning is employed to obtain generating functions for lattice statistics on Bethe lattices. It is shown that tree pruning significantly simplifies the evaluation of the generating function for the number of ways of placing k disjoint dimers on a Bethe lattice. Analytical expressions are derived for several Bethe lattices. An iterative algorithm is outlined for obtaining generating functions for placing dimers on a Bethe lattice of any valence σ and with length n. It is further shown that the method could also be applied to weighted or anisotropic lattices.

1. Introduction

One of the interesting problems in statistical physics [1] is enumerating the number of ways of placing k dimers on a lattice of N points. The problem is not only mathematically interesting but it has several physical and chemical applications such as the evaluation of the grand canonical partition function of a lattice gas, the partition function of a system of interacting ferromagnets (the Ising problem), the kinetics and thermodynamics of adsorption of diatomics on metal surfaces and in the enumeration of resonance structures of aromatic compounds. While the complete covering problem (dimer coverings) has been solved exactly for two-dimensional lattices, an analytical solution (or a generating function) for the number of ways of placing k dimers on a lattice containing N points (where $k \leqslant N/2$) is not always available for all lattices. The dimer covering problem ($k = N/2$) can be solved either by Pfaffian expansion of the associated connectivity matrix (adjacency matrix), or by the transfer matrix approach of Onsager [2]. The matching polynomial of a graph defined by Hosoya [3] generates the number of ways of placing k disjoint dimers on a lattice or graph. The

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coefficients in these polynomials generate the number of ways of placing a given number of dimers (valence bond structures) on the associated molecular graphs. Motoyama and Hosoya [4] have defined and used king polynomials, which are generating functions for the number of ways of placing a given number of non-taking kings on a chess board. Hosoya [5] has also shown the use of characteristic polynomials in physicochemical applications. Hosoya and Ohkami [6] have obtained recursion relations for matching polynomials and characteristic polynomials of certain benzenoid hydrocarbons. Gutman and coworkers [7,8] have earlier considered development of methods to compute matching polynomials.

Some years ago, Fisher and Essam [9] showed the use of Bethe lattices for percolation and cluster size problems. While Bethe lattices are not true lattices in the sense that they are topological abstractions of true lattices, exact analytical solutions can, however, be obtained for percolation and cluster size problems on these lattices. Since this development, Bethe lattices have been used in several applications. Cayley trees, which are used in a number of problems in statistical mechanics, are special cases of Bethe lattices (trivalent). In this paper, we show that exact analytical solutions for lattice statistics on Bethe lattices are possible using the method of tree pruning for the characteristic polynomials of trees developed by the present author [10] in an earlier publication. While generating functions for lattice statistics on Bethe lattices could be obtained using other methods [11], our method is applicable not only to isotropic lattices, but also weighted (edge-weighted) lattices or non-isotropic lattices. The present method is also applicable to what we call generalized Bethe lattices, for which the valences need not be the same for different lengths from the central vertex. We obtain analytical expressions for the generating functions for placing k dimers on Bethe or any tree lattices. In recent years, the development of methods for characteristic polynomials, matching polynomials, king polynomials, and their applications, has been the topic of many investigations [12–29]. Section 2 briefly reviews the basic elements of lattice statistics and its connection to matching and characteristic polynomials. Section 3 describes the tree pruning method for Bethe lattices.

2. Lattice statistics, matching polynomials and characteristic polynomials

First, we start with a brief review of the method of lattice statistics. For more details, the reader is referred to Kilpatrick [30]. Consider the grand canonical partition function of a lattice gas, which is defined by

$$\Xi = \sum_{N} \sum_{E} \Omega(N, E, V)\, e^{-\beta E}\, e^{-\alpha N} ,$$

where Ω is the number of quantum states for N molecules in volume V and energy E. One can express the above partition function in terms of the activity $z = e^{-\alpha}$ as

$$\Xi = 1 + Z_1 z + Z_2 z^2 + Z_3 z^3 + \ldots$$

with

$$Z_N = Z(N, \beta, V) = \sum_E \Omega(N, E, V) e^{-\beta E}.$$

One can write $\ln \Xi$ as

$$\ln \Xi = \sum_{j=1}^{\infty} V g_j z^j ,$$

where

$$V g_N = \frac{(-1)^{N-1}}{N} \begin{vmatrix} Z_1 & 1 & 0 & 0 \ldots . 0 \\ 2Z_2 & Z_1 & 1 & 0 \ldots . 0 \\ 3Z_3 & Z_2 & Z_1 & 1\ 0 \ldots 0 \\ NZ_n & Z_{n-1} & Z_{n-2} & \ldots . Z_1 \end{vmatrix} .$$

For lattice gases, Z_n is simply

$$Z_n = \sum_k \Omega(n, k) x^k, \qquad x = e^{-\beta\epsilon},$$

where $\Omega(n, k)$ is the number of ways of placing n particles such that they constitute k disjoint dimers on a lattice of N points. Thus, in order to compute g_N's, one needs to obtain Z_n's for various n's.

We now show the relation between the polynomial Z_n and the matching polynomials and characteristic polynomials of graphs. The matching polynomial $M_G(x)$ of a graph G is defined [3] as

$$M_G(x) = \sum_{k=0}^{m} (-1)^k P(G, k) x^{N-2k},$$

where $P(G, k)$ is the number of ways of placing k disjoint dimers on a graph. It is closely related to the Z-counting polynomial [3]

$$Q_G(x) = \sum_{k=0}^{m} P(G, k) x^k .$$

The characteristic polynomial of a graph G is defined as

$$P_G(x) = \det(A - xI),$$

where A is the adjacency matrix of the graph. The adjacency matrix element A_{ij} is one if the vertices i and j are connected; otherwise, it is zero. For any tree (a connected graph with no cycles), $P_G(x)$ and $M_G(x)$ are identical. Thus, $P_G(x)$, the characteristic polynomial of a tree, is identical with the Z_n polynomial except for the signs of the coefficients, if the associated lattice is a tree.

Bethe lattices are trees. Thus, the characteristic polynomials of these lattices are generating functions for the number of ways of placing k dimers on a lattice of N points. In the next section, we show the use of the tree pruning method for the characteristic polynomials of several Bethe lattices.

3. Tree pruning method for Bethe lattices

3.1. PRUNING BETHE LATTICES

The present author [10] developed an iterative tree pruning method for characteristic polynomials of trees. Since Bethe lattices are trees, this method should be especially of use in obtaining the characteristic polynomials or generating functions for Bethe lattices. We show here that the tree pruning method leads to analytical solutions for several Bethe lattices and, in general, provides an iterative algorithm for obtaining the characteristic polynomial of any Bethe lattice. The method is also applicable for non-isotropic lattices.

A Bethe lattice of valence σ and length n is defined simply as a tree in which each non-terminal vertex has σ neighbors (of valency σ) and there are n bonds from the central vertex to any terminal vertex. To illustrate this, fig. 1 shows a Bethe lattice of valence 4 and $n = 3$. Cayley trees are examples of Bethe lattices with valence 3. Also, trees with $\sigma = 2$ are paths of length n. Since Bethe lattices, by definition, are trees, they can be pruned into smaller trees and fragments. To illustrate, consider the Bethe lattice in fig. 1. In this lattice the terminal vertices (i.e. vertices of valence 1) can be considered as leaves. Consequently, in this particular lattice three leaves are attached to the same vertex and one can call this unit a branch. There are 12 such terminal branches for the Bethe lattice shown in fig. 1. Supposing one pruned the Bethe lattice in fig. 1 at these branch points, one would obtain the smaller tree Q_1 and a set of branches with a representative shown in the box with label T_1 in fig. 2. The pruned tree Q_1 can be called a quotient tree obtained in the first step of pruning. Equivalently, one may attach to the terminal vertices of Q_1 the closed vertex (root), a copy of T_1, to synthesize the unpruned tree in fig. 1. This graph product was formulated by the present author [31] in the context of isomer enumeration, and it was referred to as

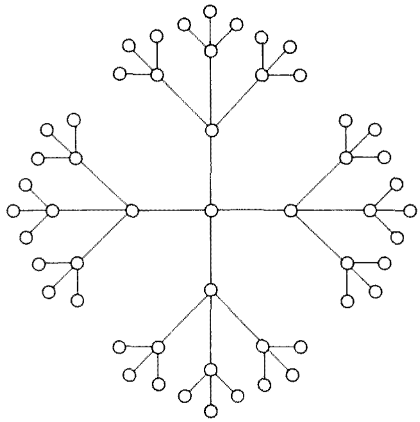

Fig. 1. A Bethe lattice with valence $\sigma = 4$ and length $n = 3$.

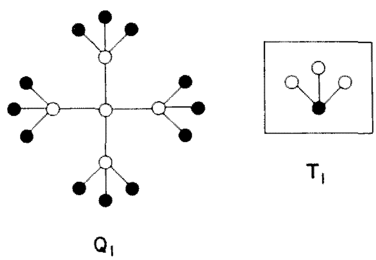

Fig. 2. The quotient tree Q_1 and a representative fragment T_1 obtained by pruning the Bethe lattice in fig. 1 at the 12 terminal branch points.

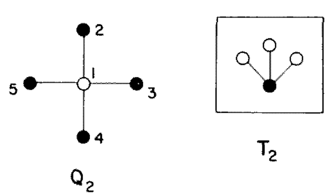

Fig. 3. The quotient tree Q_2 and the type T_2 generated by pruning the lattice in fig. 2.

a root-to-root product. The tree pruning method can be applied again to simplify this Bethe lattice further. When one prunes the Bethe lattice shown in fig. 2 at the four terminal branches, one obtains the quotient tree Q_2 and the "type" T_2 shown in fig. 3. This tree pruning method reduces a given Bethe lattice into a much smaller lattice and fragments.

3.2. TREE PRUNING AND CHARACTERISTIC POLYNOMIALS

In this section, we show that analytical expressions can be obtained for the characteristic polynomials of Bethe lattices using the tree pruning method described in subsect. 3.1.

The present author [10] showed that the characteristic polynomial of a tree can be obtained in terms of the characteristic polynomials of the pruned tree and the fragment resulting in the process of pruning. We now describe this method in order to apply it to Bethe lattices. Consider the tree in fig. 2 as an example. Our objective is, say, to obtain the characteristic polynomial of this tree which, when pruned, results in the quotient tree Q_2 and the type T_2 in fig. 3. Let the characteristic polynomial of the fragment T_2 be H_2, and let h_n denote the characteristic polynomial of a fragment containing n vertices. It can be easily seen that $h_n = \lambda^n - (n-1)\lambda^{n-2}$. Thus, $H_2 = h_4 = \lambda^4 - 3\lambda^2$. Let H_2' be the characteristic polynomial of the fragment type T_2 with the root (closed vertex) removed. In this case, when the root is deleted from T_2 it results in a disconnected graph containing three vertices. Consequently, H_2' is λ^3 for this case. Let the closed vertices (roots or branch points) of Q_2 constitute the set Y_1, and let q_{ij} be the adjacency matrix of Q_2 (i.e. $q_{ij} = 1$ if i and j are connected and 0 otherwise). Define a new adjacency matrix for Q_2 as follows:

$$A_{ij} = \begin{cases} H_2 & \text{if} \quad i = j \quad \text{and} \quad i \in Y_1 \\ \lambda & \text{if} \quad i = j \quad \text{and} \quad i \notin Y_1 \\ -H_2' q_{ij} & \text{if} \quad i \neq j \quad \text{and} \quad i \in Y_1 \\ -q_{ij} & \text{if} \quad i \neq j \quad \text{and} \quad i \notin Y_1 \end{cases} .$$

In this example, the matrix A thus constructed is shown below:

$$A = \begin{vmatrix} \lambda & -1 & -1 & -1 & -1 \\ -H_2' & H_2 & 0 & 0 & 0 \\ -H_2' & 0 & H_2 & 0 & 0 \\ -H_2' & 0 & 0 & H_2 & 0 \\ -H_2' & 0 & 0 & 0 & H_2 \end{vmatrix} .$$

The determinant of A is simply the characteristic polynomial of the tree in fig. 2 (i.e. the original unpruned tree). Of course, because of the simple nature of the matrix A, the determinant of A is easily obtained as

$$\det(A) = \lambda H_2^4 - 4H_2' H_2^3 .$$

Substituting the expression for H_2 and H_2' in the above expression, one obtains the characteristic polynomial of the tree in fig. 2 as

$$\lambda \cdot (\lambda^4 - 3\lambda^2)^4 - 4(\lambda^3)(\lambda^4 - 3\lambda^2)^3$$

$$= \lambda^{17} - 16\lambda^{15} + 90\lambda^{13} - 216\lambda^{11} + 189\lambda^9 .$$

The significance and use of the above genrating function is that the absolute value of the coefficient of λ^{17-k} in the above expression generates the number of ways of placing $k/2$ disjoint dimers on the Bethe lattice in fig. 2. For example, in fig. 2 there are 16 ways of placing one dimer, 90 ways of placing two disjoint dimers, and 216 ways of placing three dimers. Note that no more than four disjoint dimers can be placed on the lattice in fig. 2 since the coefficient of all terms with powers smaller than nine is zero. This result can easily be verified by actually attempting to place five dimers on the lattice in fig. 2.

This method of tree pruning can be iterated further. To illustrate, consider the Bethe lattice in fig. 1. In the first step, one generates the quotient tree Q_1 and the fragment type T_1. The characteristic polynomial of T_1 is $H_1 = h_4$. The characteristic polynomial $H_1' = \lambda^3$. Now the characteristic polynomial of T_2 is modified since it carries an additional branch. The characteristic polynomials H_2 and H_2' are given by:

$$H_2 = \lambda H_1^3 - 3H_1' H_1^2 ; \quad H_2' = H_1^3 .$$

The characteristic polynomial of the tree in fig. 1 is now the determinant of the matrix A, with H_2 and H_2' replaced by the above expressions.

$$\mathrm{Char}(\lambda) = \lambda H_2^4 - 4H_2' H_2^3$$

$$= \lambda(\lambda H_1^3 - 3H_1' H_1^2)^4 - 4(H_1^3)(\lambda H_1^3 - 3H_1' H_1^2)^3 .$$

Substituting $H_1 = \lambda^4 - 3\lambda^2$ and $H_1' = \lambda^3$ in the above expression one obtains the characteristic polynomial of the lattice in fig. 1. The final expression is one of the expressions in table 2 ($\sigma = 4$, $n = 3$).

3.3. ITERATIVE METHOD OF PRUNING ANY BETHE LATTICE

Consider a Bethe lattice of valence σ and length n. Let Q_i be the quotient tree generated in the ith step of pruning and T_i be the corresponding fragment type. Let H_i be the characteristic polynomial of the type T_i and H_i' be the polynomial obtained after deleting the root (branch point) in T_i. H_i and H_i' can be obtained recursively. At the nth step of pruning (where n is the length of the lattice), one obtains a simple tree whose polynomial can be obtained easily, and thus the polynomial of the original lattice we started with can be constructed recursively. For a lattice of valence σ at the first iteration, the type T_1 would contain one branch point and $(\sigma - 1)$ open vertices. Thus, the characteristic polynomial of T_1, H_1, is

$$H_1 = h_\sigma = \lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2},$$

$$H_1' = \lambda^{\sigma - 1}.$$

At the second iteration, H_2 and H_2' are expressed in terms of H_1 and H_1' as

$$H_2 = \lambda H_1^{\sigma - 1} - (\sigma - 1)H_1' H_1^{\sigma - 2},$$

$$H_2' = H_1^{\sigma - 1}.$$

(Note that H_2' is the polynomial obtained after deleting the root in T_2.) Similarly, H_3 and H_3' are expressed in terms of H_2 and H_2' as

$$H_3 = \lambda H_2^{\sigma - 1} - (\sigma - 1)H_2' H_2^{\sigma - 2},$$

$$H_3' = H_2^{\sigma - 1}.$$

Consequently, for any Bethe lattice the expressions at the ith iteration are related to the ones at the $(i - 1)$th iteration as

$$H_i = \lambda H_{i-1}^{\sigma - 1} - (\sigma - 1)H_{i-1}' H_{i-1}^{\sigma - 2},$$

$$H_i' = H_{i-1}^{\sigma - 1}.$$

Finally, at the $(n - 1)$th iteration the charachertistic polynomial of T_{n-1}, H_{n-1}, is

$$H_{n-1} = \lambda H_{n-2}^{\sigma - 1} - (\sigma - 1)H_{n-2}' H_{n-2}^{\sigma - 2},$$

$$H_{n-1}' = H_{n-2}^{\sigma - 1}.$$

Since n is the length of the lattice, the quotient tree Q_{n-1} at the $(n-1)$th iteration is a simple tree, and thus the characteristic polynomial of the lattice we started with can be obtained in terms of the determinant of the adjacency matrix of Q_n expressed in terms of H_n. This adjacency matrix is defined as

$$A_{ij} = \begin{cases} H_{n-1} & \text{if } i = j \quad \text{and} \quad i \in Y_{n-1} \\ \lambda & \text{if } i = j \quad \text{and} \quad i \notin Y_{n-1} \\ -H'_{n-1} q_{ij}^{(n-1)} & \text{if } i \neq j \quad \text{and} \quad i \in Y_{n-1} \\ -q_{ij}^{(n-1)} & \text{if } i \neq j \quad \text{and} \quad i \notin Y_{n-1} \end{cases},$$

where $q_{ij}^{(n-1)}$ is the adjacency matrix of Q_{n-1} and Y_n is the set of roots (branch points) in Q_{n-1} (there are $\sigma - 1$ vertices in Y_{n-1}). If the open vertex of Q_{n-1} carries the label 1, then the matrix A takes the form:

$$A = \begin{bmatrix} \lambda & -1 & -1 & \cdots & & -1 \\ -H'_{n-1} & H_{n-1} & 0 & \cdots & & 0 \\ -H'_{n-1} & 0 & H_{n-1} & \cdots & & 0 \\ \vdots & 0 & 0 & H_{n-1} & \cdots & 0 \\ \vdots & & & \vdots & & \\ -H'_{n-1} & \cdots 0 \cdots & 0 \cdots & 0 \cdots & 0 & H_{n-1} \end{bmatrix}.$$

The determinant of A can be easily seen to be

$$\lambda H_{n-1}^{\sigma} - \sigma H'_{n-1} H_{n-1}^{\sigma-1}.$$

The above expression is simply the characteristic polynomial of the lattice we started with. Note that the expression for H_n is obtained recursively, with H_1 defined in terms of λ and σ. One can easily see that a Bethe lattice of valence σ with length n contains υ vertices, where υ is given by

$$\upsilon = 1 + \frac{\sigma \cdot \{(\sigma - 1)^n - 1\}}{\sigma - 2}.$$

Hence, the leading power in the characteristic polynomial thus obtained would be λ^{υ}. The absolute value of the coefficient of $\lambda^{\upsilon - k}$ (k being even) gives the number of ways $k/2$ disjoining dimers can be placed on a Bethe lattice of valence σ and length n.

Table 1

The characteristic polynomials of Bethe lattices with $n = 2^{\star}$

σ	Characteristic polynomial
2	$\lambda^5 - 4\lambda^3 + 3\lambda$
3	$\lambda^{10} - 9\lambda^8 + 24\lambda^6 - 20\lambda^4$
4	$\lambda^{17} - 16\lambda^{15} + 90\lambda^{13} - 216\lambda^{11} + 189\lambda^9$
5	$\lambda^{26} - 25\lambda^{24} + 240\lambda^{22} - 1120\lambda^{20} + 2560\lambda^{18} - 2304\lambda^{16}$
6	$\lambda^{37} - 36\lambda^{35} + 525\lambda^{33} - 4000\lambda^{31} + 16\,875\lambda^{29}$
	$- 37\,500\lambda^{27} + 34\,375\lambda^{25}$

$^{\star}\sigma = 2$ is a path. $\sigma = 4$ with $n = 2$ is shown in fig. 2.

We now illustrate the use of the recursive relations thus obtained for several Bethe lattices. Consider any Bethe lattice of valence σ and $n = 2$. The characteristic polynomial of such a lattice can be obtained as a function of σ. In the first step of pruning, we obtain the following expressions:

$$H_1 = h_\sigma = \lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2},$$

$$H_1' = \lambda^{\sigma - 1}.$$

Thus, the characteristic polynomial of this lattice is given by

$$\lambda H_1^\sigma - \sigma H_1' H_1^{\sigma - 1} = \lambda(\lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2})^\sigma - \sigma\lambda^{\sigma - 1}(\lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2})^{\sigma - 1}.$$

The above expression, on simplification, yields

$$\lambda^{(\sigma - 1)^2} \cdot (\lambda^2 - \sigma + 1)^{\sigma - 1} \cdot (\lambda^2 - 2\sigma + 1).$$

Thus, an analytical expression was obtained for the characteristic polynomial of a Bethe lattice of any valence σ and $n = 2$. In table 1, we show the characteristic polynomial of a Bethe lattice with $\sigma = 3, 4, 5$ and 6 and $n = 2$.

Next, we consider Bethe lattices with valence σ and $n = 3$. The analytical expression for this case is much more complicated, as one can expect. In the first iteration, H_1 and H_1' are the same as in the earlier example. The expressions for H_2 and H_2' are:

Table 2
The characteristic polynomials of Bethe lattices with $n = 3$*

σ	Characteristic polynomial
2	$\lambda^7 - 6\lambda^5 + 10\lambda^3 - 4\lambda$
3	$\lambda^{22} - 21\lambda^{20} + 180\lambda^{18} - 816\lambda^{16} + 2112\lambda^{14}$
	$- 3120\lambda^{12} + 2432\lambda^{10} - 768\lambda^8$
4	$\lambda^{53} - 52\lambda^{51} + 1224\lambda^{49} - 17\,280\lambda^{47}$
	$+ 163\,350\lambda^{45} - 1\,092\,528\lambda^{43} + 5\,312\,700\lambda^{41}$
	$- 19\,123\,128\lambda^{39} + 50\,709\,969\lambda^{37}$
	$- 98\,021\,340\lambda^{35} + 134\,238\,060\lambda^{33}$
	$- 123\,294\,312\lambda^{31} + 68\,024\,448\lambda^{29}$
	$- 17\,006\,112\lambda^{27}$

*$\sigma = 4$ with $n = 3$ is shown in fig. 1.

$$H_2 = \lambda H_1^{\sigma - 1} - (\sigma - 1) H_1' H_1^{\sigma - 2}$$

$$= \lambda \cdot (\lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2})^{\sigma - 1} - \lambda^{\sigma - 1} \cdot (\lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2})^{\sigma - 2},$$

$$H_2' = (\lambda^\sigma - (\sigma - 1)\lambda^{\sigma - 2})^{\sigma - 1}.$$

The characteristic polynomial of the lattice is given by

$$\lambda H_2^\sigma - \sigma H_2' H_2^{\sigma - 1}.$$

In the above expression, one can substitute the appropriate expressions for H_2 and H_2'. This, on simplification, results in

$$\lambda^{(\sigma - 1)^3} \cdot (\lambda^2 - \sigma + 1)^{\sigma^2 - 2\sigma} \cdot (\lambda^2 - 2\sigma + 2)^{\sigma - 1} \cdot (\lambda^4 - 3\sigma\lambda^2 + 2\lambda^2 + \sigma^2 - \sigma).$$

In table 2, we show the resulting expressions for $\sigma = 3$ and 4. In table 3, we show the expression obtained for $\sigma = 5$ and $n = 3$. As one can see from these tables, the coefficients of terms with odd power vanish in the polynomials containing even terms and vice versa. This behavior is expected for Bethe lattices. The coefficients rise exponentially and then fall in value. For Bethe lattices, the coefficients start to vanish after a particular term, indicating that the lattice can not be covered by more than a certain number of disjoining dimers for Bethe lattices. It can be shown that the maximum number of disjoint dimers that can be placed on any Bethe lattice of valence σ and length n is given by

Table 3

The characteristic polynomials of a Bethe lattice with $\sigma = 5$ and $n = 3$

Term	Coefficient
λ^{106}	1
λ^{104}	− 105
λ^{102}	5200
λ^{100}	− 161 600
λ^{98}	3 536 640
λ^{96}	− 57 978 880
λ^{94}	739 307 520
λ^{92}	− 7 514 603 520
λ^{90}	61 892 526 080
λ^{88}	− 417 575 075 840
λ^{86}	2 323 255 394 304
λ^{84}	− 10 693 902 336 000
λ^{82}	40 729 795 624 960
λ^{80}	− 127 977 274 736 640
λ^{78}	329 618 607 308 800
λ^{77}	− 688 565 935 669 248
λ^{74}	1 147 915 909 201 920
λ^{72}	− 1 490 546 925 240 320
λ^{70}	1 452 042 543 431 680
λ^{68}	− 997 806 802 206 720
λ^{66}	431 008 558 088 192
λ^{64}	− 87 960 930 222 080

$$1 + \sigma \cdot \frac{(\sigma - 1)^n - (\sigma - 1)}{(\sigma - 1)^2 - 1} \quad \text{if } n \text{ is odd}$$

$$\sigma \cdot ((\sigma - 1)^n - 1)|(\sigma^2 - 1) \quad \text{if } n \text{ is even} .$$

It can be easily seen that the coefficient of $\lambda^{\upsilon - 2}$ (where υ is the number of vertices in the Bethe lattice) is always $\upsilon - 1$, since this gives the number of ways of covering this lattice with one dimer and hence should equal the number of bonds. Further, the signs of alternant terms change, which is in conformity with the behavior of the coefficients of characteristic polynomials of trees. Note that it is impossible to cover the entire lattice with dimers, since the constant coefficient of the characteristic polynomial of a Bethe lattice is always zero. Thus, the Pfaffian of the Bethe lattice is always zero. The Pfaffian of a lattice is defined in Montroll [1].

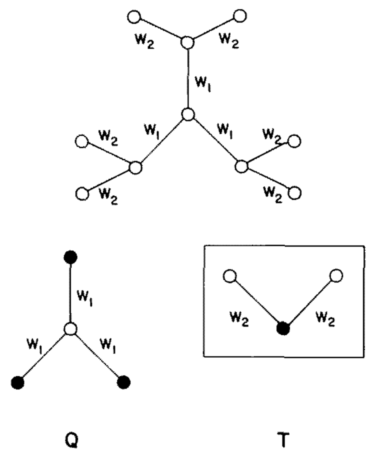

Fig. 4. A non-isotropic Bethe lattice. Q and T are the quotient graph and the fragment resulting from tree pruning. (Note that the weights w_1 and w_2 need not be equal.)

The tree pruning method described earlier is also applicable to weighted or non-isotropic lattices. For example, consider the weighted lattice graph shown in fig. 4. The weights w_1 and w_2 need not be equal (non-isotropic lattice). In this figure, we also show the quotient tree and the fragments which result in the process of pruning. The characteristic polynomial of the fragment (in the box) is given by

$$h_3 = \lambda^3 - 2\lambda w_2^2 .$$

The characteristic polynomial of the same fragment with the root deleted is given by

$$h_3' = \lambda^2 .$$

The characteristic polynomial of the quotient tree obtained using the tree pruning algorithm is given by

$$\lambda h_3^3 - 3w_1 h_3^2 h_3' .$$

When one substitutes the expressions for h_3 and h_3' for the above weighted lattice one obtains the characteristic polynomial of the lattice as

$$\lambda^{10} - \lambda^8 (6w_2^2 + 3w_1) + 12\lambda^6 (w_2^4 + w_1 w_2^2) - \lambda^4 (8w_2^6 + 12w_1 w_2^2).$$

The above example illustrates how one could obtain generating functions for the lattice statistics of non-isotropic lattices.

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